J. Fluid Mech. (1971), *vol.* 47, *part* 4, *pp.* 799–809
Printed in Great Britain

On the flow properties of a fluid between concentric spheres

By S. G. H. PHILANDER
Pierce Hall, Harvard University†

(Received 21 July 1970)

Proudman (1956) and Stewartson (1966) analyzed the dynamical properties of a fluid occupying the space between two concentric rotating spheres when the angular velocities of the spheres are slightly different, in other words, when the motion relative to a reference frame rotating with one of the spheres is due to an imposed azimuthal velocity which is symmetric about the equator. The consequences of forcing motion across the equator are explored here. Whereas the flow inside the cylinder $\mathscr{C}$ circumscribing the inner sphere and having generators parallel to the axis of rotation is similar to that which results when the driving is symmetric, the flow outside $\mathscr{C}$ is quite different. The Ekman layer on the outer sphere persists outside $\mathscr{C}$ – its dynamics is modified in the vicinity of the equator – and is instrumental in transferring fluid from one hemisphere to the other. The divergence of this Ekman layer causes slow, axial motion in the inviscid region outside $\mathscr{C}$. On $\mathscr{C}$, two shear layers of thickness $O(R^{-\frac{2}{7}})$ and $O(R^{-\frac{1}{3}})$ (where R is the Reynolds number, assumed large) remove discontinuities in the flow field and return fluid from one hemisphere to the other (rather than one Ekman layer to the other as is the case when the driving is azimuthal).

1. Introduction

The flow properties of a fluid occupying the volume between two concentric spheres which rotate about the same diameter with slightly different angular velocities have been investigated by Proudman (1956) and Stewartson (1966). In the problem they studied, relative motion is due to an imposed azimuthal velocity which is symmetric about the equator. This note concerns the flow properties of a spherical shell of fluid when motion is forced across the equator. The fluid under consideration is contained between two concentric spheres which rotate about a diameter with angular velocity Ω. Motion relative to a co-ordinate frame rotating with the spheres is due to all points on the outer sphere having a meridional velocity component V (assumed constant for mathematical convenience) and a zero azimuthal velocity component. This boundary condition may seem physically unrealistic but since it will be shown that the flow is unaffected by the presence of barriers in the form of a cylinder which has generators parallel to the axis of rotation and which has a radius smaller than that of the

† Present address: Department of Meteorology, Massachusetts Institute of Technology

inner sphere, it follows that the problem may be realized in a laboratory. It also follows that the results to be obtained, and those of Proudman (1956) and Stewartson (1966), apply equally well to the circulation in an ocean basin from which meridional barriers are absent. (The walls of such a basin intersect the bounding spheres along circles of latitude only.) The analysis of motion in an ocean basin with meridional walls when the driving velocity has both azimuthal and meridional components, and comparison of the theoretical results with the experiments of Baker & Robinson (1969), will be presented on a later occasion.

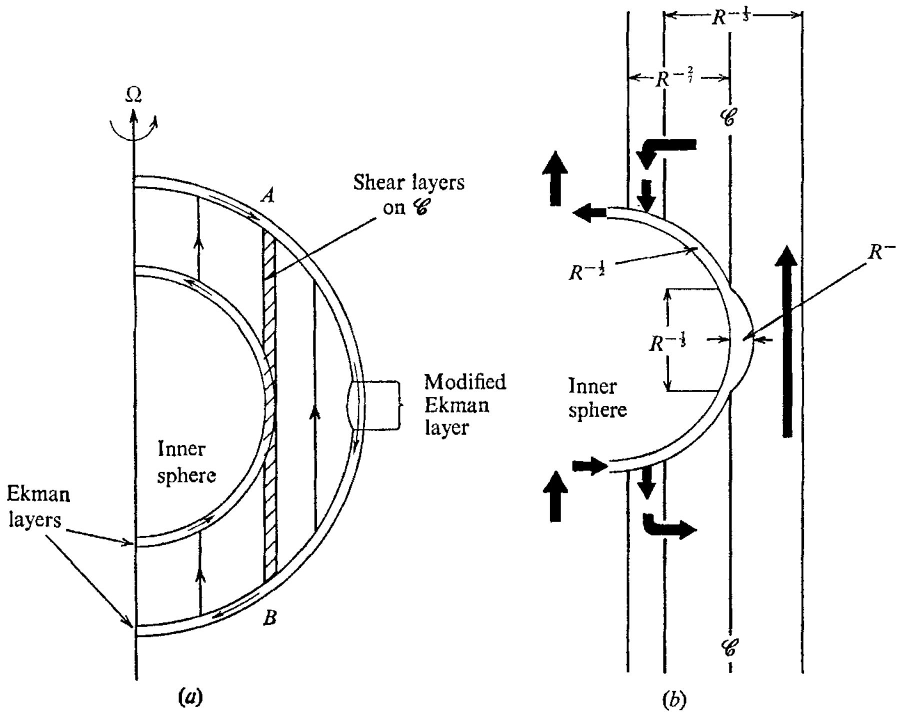

FIGURE 1. (*a*) The flow pattern. (*b*) The flow pattern in the shear layers on $\mathscr{C}$. (Drawings not to scale.)

As in the problem solved by Proudman (1956) and Stewartson (1966), the cylinder $\mathscr{C}$ which circumscribes the inner sphere and which has generators parallel to the rotation vector, separates regions of different flow properties. Inside $\mathscr{C}$ the flow in the event of meridional driving is identical to the flow which results when the driving is zonal. In the northern hemisphere fluid remains on a cylinder coaxial with $\mathscr{C}$ while moving from the Ekman layer on the inner sphere (which has a northward transport) to the Ekman layer on the outer sphere (which has a southward transport). The axial velocity component is $O(R^{-\frac{1}{2}})$, the azimuthal velocity component $O(1)$. (Geophysical conventions have been adopted and it is assumed that the $O(1)$ driving velocity is southward. The Reynolds number $R = a^2\Omega/\nu$ where a is the radius of the inner sphere and ν is the coefficient of viscosity, is assumed large.) In the southern hemisphere the azimuthal velocity component is reversed, but the axial and meridional velocities are in the same direction as in the northern hemisphere. (See figure 1 (*a*).)

Whereas there is rigid body rotation outside $\mathscr{C}$ when the driving is azimuthal, the Ekman layer on the outer sphere persists outside $\mathscr{C}$ when the driving is meridional and its divergence causes $O(R^{-\frac{1}{2}})$ axial motion, but no azimuthal motion, in the inviscid region outside $\mathscr{C}$. The extra-equatorial Ekman layer dynamics which is characterized by a balance between viscous stresses and the Coriolis force associated with the radial component of the rotation vector, breaks down when $\theta \sim \frac{1}{2}\pi + O(R^{-\frac{1}{5}})$, θ being the angle of colatitude. At that latitude, the Coriolis force associated with the meridional component of the rotation vector becomes important. It is found that this region of higher-order dynamics, the modified Ekman layer, also removes a singularity which the axial velocity component of the inviscid flow has at the intersection of the outer sphere and the equator.

Since the transport of the Ekman layer on the outer sphere is continuous across the intersection of $\mathscr{C}$ and the outer sphere, and is in the same direction at both A and B (see figure 1 (*a*)), there is a net transfer of fluid from one hemisphere to the other. Though it seems as if the Ekman layer on the inner sphere could return this fluid, the return flow is actually via shear layers on $\mathscr{C}$. These shear layers also remove discontinuities in the flow fields. As in the case investigated by Stewartson (1966) shear layers of width $O(R^{-\frac{2}{7}})$ and $O(R^{-\frac{1}{3}})$ smooth out discontinuities in the gradient of the azimuthal velocity and the velocity component radially out from the axis, respectively. It is the latter layer which transports fluid across the equator, losing it to the $O(R^{-\frac{2}{7}})$ layer (in the northern hemisphere) which returns the fluid to the Ekman layer on the inner sphere (see figure 1 (*b*)). A shear layer of width $O(R^{-\frac{1}{4}})$ which is necessary to remove a discontinuity in the azimuthal velocity when the driving is zonal is unnecessary here because there is no zonal motion outside $\mathscr{C}$. The role of the modified Ekman layer on the inner sphere is secondary.

The linear theory is valid provided $(V/a\Omega)\, R^{-\frac{2}{5}} \ll 1$. For values of V exceeding this limit, the dynamics of the modified equatorial Ekman layer breaks down due to non-linear effects.

2. The model

Let the radii of the inner and outer spheres be a and αa respectively and let Ω be the angular velocity of both spheres about a diameter which is described by $\theta = 0$ in spherical polar co-ordinates (ar, θ, ϕ). Let (Vu, Vv, Vw) be the corresponding velocity components (see figure 2) measured relative to the co-ordinate system which rotates with the spheres. Relative motion is due to all points on the outer sphere having a constant meridional velocity V.

Since the dynamical variables must be independent of ϕ (by symmetry), the velocity components may be expressed in terms of two functions ψ and χ.

$$u = \frac{1}{r^2 \sin\theta}\psi_\theta, \quad v = -\frac{1}{r\sin\theta}\psi_r, \quad w = \frac{\chi}{r\sin\theta}. \tag{2.1}$$

If non-linear terms are neglected, then the equations of motion may be written

$$2\left(\frac{\partial \chi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \chi}{\partial \theta}\sin\theta\right) = \frac{1}{R}D^4\psi, \tag{2.2}$$

$$-2\left(\frac{\partial \chi}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial \psi}{\partial \theta}\sin\theta\right) = \frac{1}{R}D^2\chi, \tag{2.3}$$

where
$$D^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

and $R = a^2\Omega/\nu \gg 1$ is the Reynolds number of the flow. These equations must be solved subject to the conditions that

$$\partial\psi/\partial r = \psi = \chi = 0 \quad \text{on} \quad r = 1, \tag{2.4}$$

$$\psi = \chi = 0, \qquad \partial\psi/\partial r = -\alpha\sin\theta \quad \text{on} \quad r = \alpha. \tag{2.5}$$

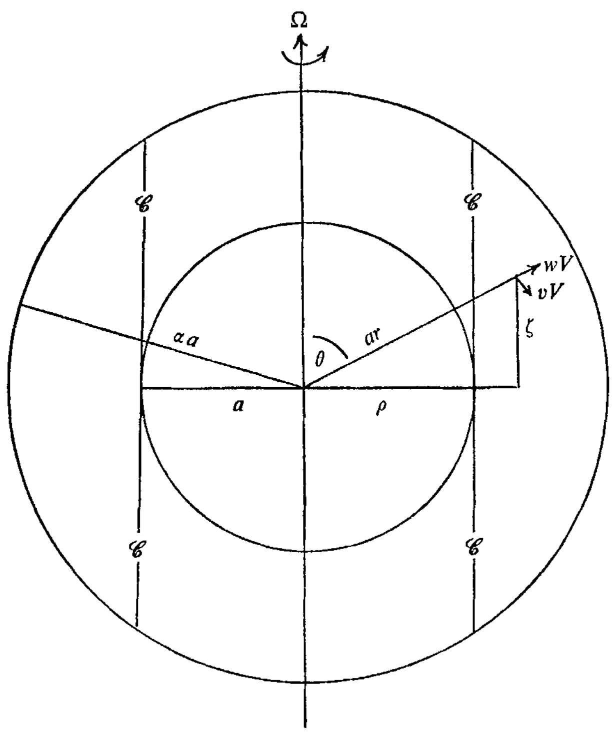

FIGURE 2. Notation.

For future purposes we record the equations of motion in cylindrical polar co-ordinates (ρ, ϕ, ζ) where

$$\rho = r\sin\theta, \quad \zeta = r\cos\theta. \tag{2.6}$$

The governing equations are

$$\frac{1}{R}D^4\psi = 2\frac{\partial\chi}{\partial\zeta}, \quad \frac{1}{R}D^2\chi = -2\frac{\partial\psi}{\partial\zeta}, \tag{2.7}$$

where
$$D^2 = \frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{\partial^2}{\partial\zeta^2}.$$

3. The flow inside $\mathscr{C}$

If it is assumed that viscous stresses are negligible in the main body of the fluid, then the functions ψ and χ must satisfy (2.7) with $R = \infty$ so that

$$\chi = \chi_0(\rho), \quad \psi = \psi_0(\rho). \tag{3.1}$$

This solution fails to satisfy the boundary conditions on the spheres, so that boundary-layer corrections must be invoked in the vicinity of the spheres. The dynamics of these Ekman layers is described by the equations

$$\frac{\partial^4 \psi}{\partial r^4} = 2R\cos\theta\frac{\partial \chi}{\partial r}, \quad \frac{\partial^2 \chi}{\partial r^2} = -2R\cos\theta\frac{\partial \psi}{\partial r}. \tag{3.2}$$

It may be shown (Proudman 1956) that, in the northern hemisphere where $|\theta| < \frac{1}{2}\pi$, χ_0 and ψ_0 must satisfy the compatibility conditions

$$\chi_0(\rho) = 2R^{\frac{1}{2}}(1-\rho^2)^{\frac{1}{4}}\psi_0(\rho), \tag{3.3}$$

$$\rho - \chi_0(\rho) = 2R^{\frac{1}{2}}(1-(\rho^2/\alpha^2))^{\frac{1}{4}}\psi_0(\rho), \tag{3.4}$$

by studying the Ekman layers on the inner and outer spheres respectively. The same equations with χ_0 replaced by $-\chi_0$ are the relevant ones in the southern hemisphere. It follows that ψ_0 is an even function of latitude, χ_0 an odd function of latitude.

Inside $\mathscr{C}$ where both (3.3) and (3.4) must be satisfied

$$\psi_0(\rho) = (\rho/2R^{\frac{1}{2}})\,[(1-\rho^2)^{\frac{1}{4}} + (1-(\rho^2/\alpha^2))^{\frac{1}{4}}]^{-1}, \tag{3.5}$$

$$\chi_0(\rho) = \rho(1-\rho^2)^{\frac{1}{4}}\,[(1-\rho^2)^{\frac{1}{4}} + (1-(\rho^2/\alpha^2))^{\frac{1}{4}}]^{-1}, \tag{3.6}$$

provided $\rho < 1$ and $|\theta| < \frac{1}{2}\pi$. The flow may be described as follows. In the inviscid region fluid remains on a cylinder coaxial with $\mathscr{C}$ while it gets transferred from the Ekman layer on the inner sphere to the Ekman layer on the outer sphere. At a given value of ρ, the transports of the Ekman layers are equal and opposite, that of the Ekman layer on the outer sphere being away from the axis of rotation. This description applies to the northern hemisphere provided the driving is southwards. Since the meridional velocity component is symmetric about the equator, the meridional transports of the respective Ekman layers are, in the southern hemisphere, in the same direction as in the northern hemisphere. However, the axial flow in the southern hemisphere is from the Ekman layer on the outer sphere to the Ekman layer on the inner sphere.

4. The flow outside $\mathscr{C}$

Outside $\mathscr{C}$ (3.3) ceases to be relevant. Because χ_0 is an odd function of latitude

$$\left.\begin{aligned} &\chi_0 = 0 \quad \text{at} \quad \zeta = 0, \quad \rho > 1, \\ &\chi_0 \equiv 0, \quad \psi_0(\rho) = \frac{\rho}{2R^{\frac{1}{2}}}\left(1-\frac{\rho^2}{\alpha^2}\right)^{-\frac{1}{4}} \quad \text{when} \quad \rho > 1. \end{aligned}\right\} \tag{4.1}$$

The only motion in the inviscid region outside $\mathscr{C}$ is parallel to the axis of rotation.

Note that the expression for $\psi_0(\rho)$ is singular at the intersection of the equatorial plane and the outer sphere: at $\rho = \alpha$.

It has been shown (Stewartson 1966) that the Ekman layer dynamics, described by (3.2), breaks down when $\theta - \frac{1}{2}\pi = O(R^{-\frac{1}{5}})$. The terms $-(2/r)\sin\theta\chi_\theta$ and $(2/r)\sin\theta\,\psi_\theta$ in (2.2) and (2.3) respectively become significant at this latitude. In other words, the Coriolis force associated with the meridional component of the rotation vector cannot be neglected close to the equator. Following Stewartson (1966), we remove non-dimensional parameters from the modified Ekman layer equations by writing

$$\chi = \chi^*, \quad \psi = R^{-\frac{2}{5}}\psi^*, \quad \tfrac{1}{2}\pi - \theta = R^{-\frac{1}{5}}\mathscr{H}, \quad r = \alpha - \lambda R^{-\frac{2}{5}},\dagger \tag{4.2}$$

in which case the equations read as follows:

$$\frac{\partial^4\psi^*}{\partial\lambda^4} = 2\left(-\mathscr{H}\frac{\partial\chi^*}{\partial\lambda} + \frac{1}{\alpha}\frac{\partial\chi^*}{\partial\mathscr{H}}\right); \quad \frac{\partial^2\chi^*}{\partial\lambda^2} = -2\left(-\mathscr{H}\frac{\partial\psi^*}{\partial\lambda} + \frac{1}{\alpha}\frac{\partial\psi^*}{\partial\mathscr{H}}\right). \tag{4.3}$$

Note that $\rho = \alpha - R^{-\frac{2}{5}}(\lambda + \frac{1}{2}\alpha\mathscr{H}^2)$ so that the breakdown of the Ekman layer dynamics takes place when

$$\rho - \alpha = O(R^{-\frac{2}{5}}).$$

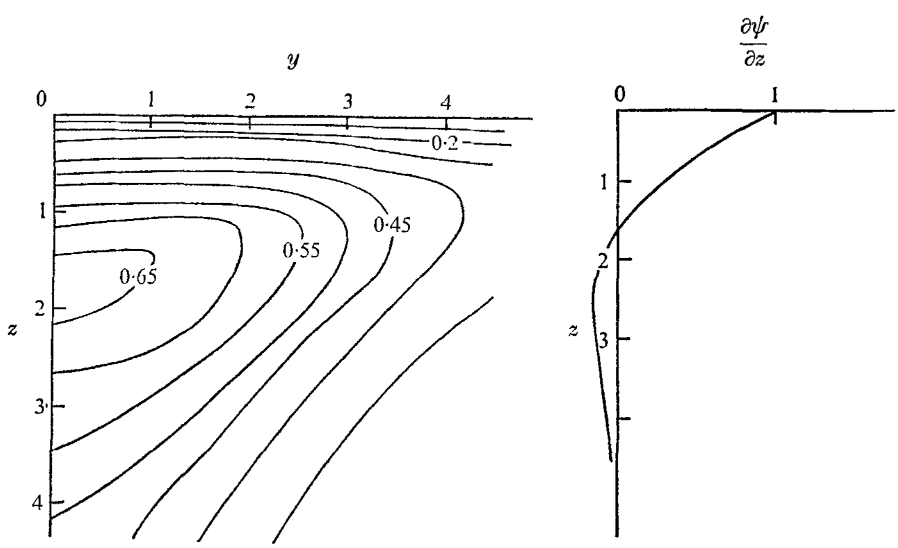

FIGURE 3. (*a*) Streamlines in the modified Ekman layer. (*b*) The meridional velocity profile at the equator in the modified Ekman layer.

It follows from (4.1) that in this region

$$\psi_0(\rho) \sim O(R^{-\frac{2}{5}}).$$

Hence the singularity of ψ_0 is consistent with the match the inviscid flow makes with flow in the modified Ekman layer.

† The parameter range for which the depth of this modified Ekman layer is comparable to $\alpha - 1$ (i.e. the parameter range for which viscous effects are important throughout the fluid, near the equator) has been investigated by Carrier (1965) and Philander (1970).

Equations (4.3) have been solved numerically (using a method given in the appendix) subject to the boundary conditions that

$$\text{(i)} \quad \chi^* = \psi^* = 0, \quad \partial\psi^*/\partial\lambda = \alpha \quad \text{on} \quad \lambda = 0; \tag{4.4}$$

$$\text{(ii)} \quad \chi^*, \psi^*, \psi^*_\lambda \to 0 \quad \text{as} \quad \lambda \to \infty; \tag{4.5}$$

$$\text{(iii)} \quad \chi^* = \partial\psi^*/\partial\mathscr{H} = 0 \quad \text{on} \quad \mathscr{H} = 0; \tag{4.6}$$

$$\text{(iv)} \quad \chi^* \sim 2(\tfrac{1}{2}\alpha)^{\frac{5}{4}}\lambda^{-\frac{1}{4}}\mathscr{H}^{\frac{1}{2}}e^{-\eta}\cos\eta - \alpha e^{-\eta}\cos\eta; \tag{4.7}$$

$$\psi^* \sim (\tfrac{1}{2}\alpha)^{\frac{5}{4}}\lambda^{-\frac{1}{4}}[1 - e^{-\eta}(\cos\eta + \sin\eta)] + \alpha\mathscr{H}^{-\frac{1}{2}}e^{-\eta}\sin\eta, \tag{4.8}$$

when $\mathscr{H} \gg 1$, where $\eta = \lambda\mathscr{H}^{\frac{1}{2}}$. (This is simply the solution to the extra-equatorial Ekman layer equations.) The substitution

$$(\lambda, \mathscr{H}, \chi^*, \psi^*) \to (\alpha^{\frac{1}{5}}z, \alpha^{-\frac{2}{5}}y, \tilde{\chi}\alpha, \alpha^{\frac{6}{5}}\tilde{\psi})$$

eliminates α from equations (4.3) to (4.8). Figure 3 (*a*) depicts streamlines in the y, z plane and figure 3 (*b*) the meridional velocity profile at the equator.

5. Shear layers on $\mathscr{C}$

Whereas the shear layers analyzed by Stewartson (1966) transport fluid from one Ekman layer to the other, it is not clear that it is at all necessary for shear layers to transport fluid in the case under consideration. Though it is conceivable that the fluid could return from the lower to the upper hemisphere in the Ekman layer and modified Ekman layer on the inner sphere, it will be seen that the return is via shear layers on $\mathscr{C}$ which also remove discontinuities in the fields.

If it is assumed that in the shear layers differentiation with respect to ρ has a magnifying effect when $R \gg 1$, but that differentiation with respect to ζ does not have such an effect, then the governing equations (2.7) reduce to

$$\partial 4\psi/\partial\rho^4 = 2R\,\partial\chi/\partial\zeta, \quad \partial^2\chi/\partial\rho^2 = -2R\,\partial\psi/\partial\zeta. \tag{5.1}$$

These equations must be solved subject to the conditions that

$$\chi \to 0, \quad \psi \to \frac{1}{2R^{\frac{1}{2}}}\left(1 - \frac{1}{\alpha^2}\right)^{-\frac{1}{4}} \quad \text{as} \quad \rho - 1 \to \infty, \tag{5.2}$$

$$\chi \to 2^{\frac{1}{4}}\left(\frac{1-\rho}{1-(1/\alpha^2)}\right)^{\frac{1}{4}}, \quad \psi \to \frac{1}{2R^{\frac{1}{2}}}\left(1 - \frac{1}{\alpha^2}\right)^{-\frac{1}{4}} \quad \text{as} \quad \rho - 1 \to -\infty. \tag{5.3}$$

It will be seen that the limits $\rho - 1 \to \mp\infty$ should be interpreted as

$$1 - \rho \gg O(R^{-\frac{2}{7}}) \quad \text{and} \quad \rho - 1 \gg O(R^{-\frac{1}{3}})$$

respectively. The additional boundary conditions at the intersections of the shear layers and the Ekman layers are simply the compatibility conditions (3.3) and (3.4).

$$1 - \chi = 2R^{\frac{1}{2}}(1 - (1/\alpha^2))^{\frac{1}{4}}\psi \quad \text{at} \quad \zeta = (\alpha^2 - 1)^{\frac{1}{2}}, \tag{5.4}$$

$$\chi = 2R^{\frac{1}{2}}[(1 - \rho^2)]^{\frac{1}{4}}\,\psi \quad \text{at} \quad \zeta = 0 \quad \text{provided}\ \rho < 1. \tag{5.5}$$

These relations are valid provided $\frac{1}{2}\pi - \theta > O(R^{-\frac{1}{5}})$. Finally,

$$\psi_\zeta = 0 \quad \text{at} \quad \zeta = 0 \quad \text{provided} \quad \rho > 0 \tag{5.6}$$

is necessary from symmetry considerations.

Though neither χ nor ψ is discontinuous on $\mathscr{C}$, χ_ρ is. It may be shown that a shear layer of width $O(R^{-\frac{1}{3}})$ can remove a discontinuity in χ_ρ only if it is smaller than $O(R^{\frac{1}{6}})$. The actual discontinuity is $O(R^{\frac{1}{4}})$ so that an $O(R^{-\frac{2}{7}})$ layer is necessary. This layer, however, introduces a discontinuity in ψ which may be removed by the introduction of an $O(R^{-\frac{1}{3}})$ layer. Though the $O(R^{-\frac{2}{7}})$ layer does not receive fluid from the Ekman layer on the outer sphere, it loses fluid to the Ekman layer on the inner sphere (in the northern hemisphere). This fluid is obtained from the $O(R^{-\frac{1}{3}})$ layer which accepts fluid from the $O(R^{-\frac{2}{7}})$ layer in the southern hemisphere and transports it across the equator (see figure 1(*b*)).

(i) *The layer of width* $O(R^{-\frac{2}{7}})$

The analysis is essentially the same as that presented by Stewartson (1966). Assuming that the first of equations (5.1) may be approximated by $\chi_\zeta = 0$ (this may be verified later), one may deduce from (5.1), (5.4) and (5.5) that

$$\chi = Bs^{\frac{1}{2}}K_{\frac{4}{7}}(\tfrac{8}{7}s^{\frac{7}{8}}) + 2^{\frac{2}{7}}(\alpha^2-1)^{-\frac{5}{28}}R^{-\frac{1}{14}}\mathscr{G}(s),$$

where B is an arbitrary constant, $K_{\frac{4}{7}}$ is the Bessel function of order $\frac{4}{7}$, of the second kind with imaginary argument.

$$s = (1-\rho)\,[R^2/2(\alpha^2-1)^2]^{\frac{1}{7}}$$

and $\mathscr{G}'' - s^{-\frac{1}{4}}\mathscr{G} = -1$ with $\mathscr{G}(0) = 0$ and $\mathscr{G} - s^{\frac{1}{4}}$ bounded as $s \to \infty$. The corresponding value for ψ is

$$\psi = \frac{(\alpha^2-1)^{\frac{1}{2}} - \zeta}{2R^{\frac{1}{2}}(\alpha^2-1)^{\frac{1}{2}}}\,[2(1-\rho)]^{-\frac{1}{4}} + \frac{\zeta(1-\chi)\,\alpha^{\frac{1}{2}}}{2R^{\frac{1}{2}}(\alpha^2-1)^{\frac{3}{4}}}.$$

Choosing $B = 0$ ensures that both χ and χ_ρ are continuous on $\mathscr{C}$. However, ψ is discontinuous so that another boundary layer, an $O(R^{-\frac{1}{3}})$ shear layer, has to be introduced. It is not possible to do without the $O(R^{-\frac{2}{7}})$ layer, which removes the discontinuity in χ_ρ, for it may be shown that the $O(R^{-\frac{1}{3}})$ layer can remove discontinuities in χ_ρ only if they are smaller than $O(R^{\frac{1}{6}})$.

Note that the value of ψ is constant (to $O(R^{-\frac{1}{2}})$) at $\zeta = (\alpha^2-1)^{\frac{1}{2}}$ so that the shear layer does not lose fluid to, or gain fluid from, the Ekman layer on the outer sphere. There is, however, a loss of fluid to the Ekman layer on the inner sphere in the northern hemisphere. Since

$$\psi \to \frac{\zeta\alpha^{\frac{1}{2}}}{2R^{\frac{1}{2}}(\alpha^2-1)^{\frac{3}{4}}} \quad \text{as} \quad \rho \to 1- \tag{5.7}$$

there is a flux away from $\mathscr{C}$ to compensate for this loss.

(ii) *The layer of width* $O(R^{-\frac{1}{3}})$

The discontinuity in ψ is $O(R^{-\frac{1}{2}})$ (see (5.7) and (5.2)), prompting the substitution

$$\psi = R^{-\frac{1}{2}}\overline{\psi}, \quad \chi = R^{-\frac{1}{6}}\overline{\chi}, \quad \rho - 1 = R^{-\frac{1}{3}}\nu.$$

The equations describing motion in the $O(R^{-\frac{1}{3}})$ layer are

$$\partial^4\bar{\psi}/\partial\nu^4 = 2\,\partial\bar{\chi}/\partial\zeta, \quad \partial^2\bar{\chi}/\partial\nu^2 = -2\,\partial\bar{\psi}/\partial\zeta$$

and must be solved subject to the conditions that

(i) $\bar{\psi} = \dfrac{\alpha^{\frac{1}{2}}}{2(\alpha^2-1)^{\frac{1}{4}}}$ at $\zeta = (\alpha^2-1)^{\frac{1}{2}}$;

(ii) $\bar{\psi} = 0$ at $\zeta = 0$ provided $\nu < 0$;

(iii) $\bar{\psi}_\zeta = 0$ at $\zeta = 0$ provided $\nu > 0$;

(iv) $\bar{\psi} \to \dfrac{\zeta\alpha^{\frac{1}{2}}}{2(\alpha^2-1)^{\frac{3}{4}}}$, $\bar{\chi} \to 0$ as $\nu \to -\infty$;

(v) $\bar{\psi} \to \dfrac{\alpha^{\frac{1}{2}}}{2(\alpha^2-1)^{\frac{1}{4}}}$, $\bar{\chi} \to 0$ as $\nu \to +\infty$.

By taking Fourier transforms, employing the Wiener–Hopf technique (Noble 1958) and keeping in mind that

$$\lambda^3 b \coth \lambda^3 b = K_+(\lambda)\,K_-(\lambda),$$

where
$$K_+(\lambda) = \pi^{\frac{1}{2}}\Gamma(1-i(b\lambda^3/\pi))/\Gamma(\tfrac{1}{2}-i(b\lambda^3/\pi))$$

and
$$K_-(\lambda) = K_+(-\lambda),$$

one can show that

$$\bar{\psi} = \frac{1}{2(1-(1/\alpha^2))^{\frac{1}{4}}} - 1 + \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda\nu}\frac{\cosh\lambda^3\zeta}{\epsilon - i\lambda}\frac{K_-(-i\epsilon)}{K_-(\lambda)} - \frac{\sinh\lambda^3\zeta}{\epsilon-i\lambda}\frac{K_+(\lambda)}{\lambda^3(\alpha^2-1)^{\frac{1}{2}}}K_+(i\epsilon)\,d\lambda, \tag{5.8}$$

as $\epsilon \to 0$ and $\epsilon > 0$. The value of ψ being constant when $\zeta = (\alpha^2-1)^{\frac{1}{2}}$ and when $\zeta = 0$, $\nu < 0$, no fluid is lost to either Ekman layer by the $O(R^{-\frac{1}{3}})$ layer. Yet a volume of fluid equal to $\frac{1}{2}R^{-\frac{1}{2}}(1-(1/\alpha^2))^{-\frac{1}{4}}$ is lost to the $O(R^{-\frac{2}{7}})$ layer in the northern hemisphere. This fluid must necessarily cross the equator from the southern hemisphere in the $O(R^{-\frac{1}{3}})$ layer. That this is indeed the case may be verified by showing that $\bar{\psi} \to 0$ as $\nu \to 0$ on $\zeta = 0$. The integral in (5.8) may be evaluated in terms of the residues at the poles of the various Γ functions:

$$\bar{\psi} \sim -\frac{\alpha^{\frac{1}{2}}}{2(\alpha^2-1)^{\frac{1}{4}}}\left[1-\frac{F(1)}{\pi^{\frac{3}{2}}}\right] \quad \text{as} \quad \nu \to 0, \nu > 0 \text{ and } \zeta = 0,$$

where
$$F(x) = \sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}\frac{x^{n+\frac{1}{2}}}{n+\frac{1}{2}}.$$

It follows that

$$F'(x) = \sum_0^\infty \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}x^{n-\frac{1}{2}}.$$

But
$$F(0) = 0,$$

therefore
$$F(x) = \Gamma(\tfrac{1}{2})\,2\arcsin x,$$
$$F(1) = \pi^{\frac{3}{2}},$$
$$\bar{\psi} = 0 \quad \text{as} \quad \nu \to 0+ \quad \text{on} \quad \zeta = 0.$$

Note that the volume of fluid crossing the equator in the $O(R^{-\frac{1}{3}})$ layer is exactly equal to the volume which the Ekman layer on the outer sphere transports across the points A and B (see figure 1 (a)).

It is the function of the modified Ekman layer on the inner sphere to bring the $O(R^{-\frac{1}{6}})$ azimuthal velocity to zero on the inner sphere when $\frac{1}{2}\pi-\theta < O(R^{-\frac{1}{6}})$. The velocities in this modified Ekman layer, and its transport, are $O(R^{-\frac{1}{6}})$ relative to that of the modified Ekman layer on the outer sphere so that the role of the modified Ekman layer on the inner sphere is secondary.

The support of the Office of Naval Research under contract number N 00014-67-A-0298-0011 to Harvard University is gratefully acknowledged.

Appendix. A numerical method for solving the equations describing flow in the modified Ekman layer

If equations (4.3) are converted into time-dependent equations, they read

$$\tilde{\psi}_{tzz} = \tilde{\psi}_{zzzz} - 2(-y\tilde{\chi}_z + \tilde{\chi}_y),$$
$$\tilde{\chi}_t = \tilde{\chi}_{zz} + 2(-y\tilde{\psi}_z + \tilde{\psi}_y),$$

where t represents time non-dimensionalized with respect to $(a^2/\nu\Omega^4)^{\frac{1}{5}}$.

The method of solution exploits the fact that the equations of motion are parabolic in time so that initial conditions are sufficient to calculate the fields at a later time. If the driving velocity is imposed impulsively at time $t = 0$, the asymptotic solution as $t \to \infty$ should correspond to the steady-state solution.

The following is the finite difference scheme used.

$$y = j\Delta y, \quad z = k\Delta z, \quad t = m\Delta t,$$
$$\tilde{\chi}(y,z,t) = \chi^{m}_{jk}, \quad \tilde{\psi}(y,z,t) = \psi^{m}_{jk},$$
$$\chi^{m}_{jk} = \chi^{m-1}_{jk} + \Delta t[(\chi^{m-1}_{jk+1} - 2\chi^{m-1}_{jk} + \chi^{m-1}_{jk-1})/\Delta z^2] + \frac{j\Delta y\Delta t}{\Delta z}(\psi^{m-1}_{jk+1} - \psi^{m-1}_{jk-1}) - \frac{\Delta t}{\Delta y}(\psi^{m-1}_{j+1k} - \psi^{m-1}_{j-1k})$$

The equation for $(\tilde{\psi}_{zz})^{m}_{jk}$ is similar. The boundary conditions (4.4)–(4.8) are the relevant ones. The domains of z and y were chosen as $0 < z < 4$, $0 < y < 4$. Increasing the upper limit did not affect the results significantly. The value 0·1 was adopted for Δy and Δz. Computations were repeated for smaller values but the results did not change appreciably. Steady-state conditions were attained after a non-dimensional time of 4 units.

REFERENCES

Baker, D. J. & Robinson, A. R. 1969 A laboratory model for the general-ocean circulation. *Phil. Trans. Roy. Soc.* A **265**, 1168.

Carrier, G. F. 1965 Some effects of stratification and geometry in rotating fluids. *J. Fluid Mech.* **23**, 145–172.

NOBLE, B. 1958 *The Wiener–Hopf technique.* Pergamon.

PHILANDER, S. G. 1970 The equatorial dynamics of a shallow, homogeneous ocean. *Geophysical Fluid Dynamics* (in press).

PROUDMAN, I. 1956 The almost rigid rotation of viscous fluid between concentric spheres. *J. Fluid Mech.* **1**, 505–516.

STEWARTSON, K. 1966 On almost rigid rotations. Part 2. *J. Fluid Mech.* **26**, 131–144